THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 9

Update at 24/4/2017:

- In Q1, the exact value of $\rho$ can be found out.
- In Q3, the imaginary part of $g(z)$ should be $\operatorname{Im} g(i) = \frac{ad-bc}{c^2+d^2}$ instead of $\operatorname{Im} g(i) = ad - bc$.

1 (a) The required linear fractional transformation is implicitly given by

$$(z, -\rho, \rho, 1) = (w, 0, \frac{2}{3}, 1)$$

The explicit form of the transformation is given by

$$w = f(z) = \frac{(4\rho - 2\rho^2)z + 2\rho^2 - 2\rho}{2(2\rho - 1)z - 2(\rho - 1)}$$

To find out the value of $\rho$, since $f$ maps the straight line containing $-1, 0, \frac{2}{3}, 1$ to the straight line containing $-1, -\rho, \rho, 1$, we must have $f(-1) = -1$. This equation is equivalent to

$$\rho^2 - 3\rho + 1 = 0$$

Since $\rho < 1$, we have $\rho = \frac{3-\sqrt{5}}{2}$.

(b) Consider the function

$$g(z) = g(x, y) = \frac{1}{\ln(\frac{1}{\rho})} \ln \frac{\sqrt{x^2 + y^2}}{\rho}$$

defined on the annulus of the form $\rho < |z| < 1$. Note that locally it is the real part of the analytic function

$$h(z) = \frac{1}{\ln(\frac{1}{\rho})} \log_\alpha \frac{z}{\rho}$$

for some $\alpha$. Hence it is harmonic (you may also verify it by direct computation). Furthermore, the function $u$ satisfies the properties that $u|_{|z|=\rho} = 0$ and $u|_{|z|=1} = 1$.

Therefore, we consider the function $u$ on $D$ defined by

$$u(z) = u(x, y) = g(f(z))$$

This function is harmonic since it is the composition of a harmonic function $g$ with an analytic function $f$. Also this function satisfies the desired properties.

2 (a) Note that for any $(u, v) \in \mathbb{R}$, we have

$$h_{uu} + h_{vv} = -e^{-v} \sin u + e^{-v} \sin u = 0$$

Hence $h$ is a harmonic function.

(b) Since $h$ is harmonic and $f(z) = z^2$ is an analytic function on $\{(x, y)|x, y > 0\}$, the function

$$h(f(x, y)) = h(x^2 - y^2, 2xy) = e^{-2xy}\sin(x^2 - y^2)$$

is a harmonic function.

3 First of all, recall that the map $\phi(z) = i\dfrac{1-z}{1+z}$ maps $\{|z| < 1\}$ conformally onto $\{x + iy|y > 0\}$. Then $g$ is a conformal self-map of upper half plane if and only if $\phi^{-1} \circ g \circ \phi$ is a conformal self-map of $\{|z| < 1\}$. In particular, since every conformal self-map of $\{|z| < 1\}$ is linear fractional transformation, $g$ must also be a linear fractional transformation.

Let $g(z) = \dfrac{az + b}{cz + d}$ for some $a, b, c, d \in \mathbb{C}$. Note that $g$ maps the x-axis onto the x-axis. Pick any three real numbers $x_1, x_2$ and $x_3$ such that $g(x_i) \neq \infty$ for $i = 1, 2, 3$. Then $g(z)$ can be implicitly expressed as

$$(g(z), g(x_1), g(x_2), g(x_3)) = (z, x_1, x_2, x_3)$$

From this we can see that $a, b, c$ and $d$ are real numbers.

Furthermore, one can verify that $\operatorname{Im} g(i) = \frac{ad-bc}{c^2+d^2}$. Therefore we must have $ad - bc > 0$. Finally, we can normalize $g(z)$ to be

$$g(z) = \frac{\frac{a}{\sqrt{ad-bc}}z + \frac{b}{\sqrt{ad-bc}}}{\frac{c}{\sqrt{ad-bc}}z + \frac{d}{\sqrt{ad-bc}}} = \frac{Az + B}{Cz + D}$$

such that $AD - BC = 1$.

4 Recall that the map $\phi(z) = i\dfrac{1-z}{1+z}$ maps $\{|z| < 1\}$ conformally onto $\{x + iy|y > 0\}$. Therefore, the inverse function $\phi^{-1}(z) = \dfrac{i-z}{i+z} = -1 + \dfrac{2i}{i+z}$ maps $\{x + iy|y > 0\}$ conformally onto$\{|z| < 1\}$. Moreover, $\phi^{-1}(i) = 0$.

Now suppose the function $f$ satisfies the property that $f(0) = i$. Since $\operatorname{Im}(f) > 0$, the map $\phi^{-1} \circ f$ satisfies $|f(z)| \leq 1$. Moreover, $\phi^{-1}(f(0)) = \phi^{-1}(i) = 0$. As a result, by Schwartz's Lemma,

$$\begin{aligned} & |\frac{d}{dz}\phi^{-1}(f(0))| \leq 1 \\ \implies \quad & |2i\frac{-1}{(f(0)+i)^2}f'(0)| \leq 1 \\ \implies \quad & |f'(0)| \leq 2 = 2\operatorname{Im} f(0) \end{aligned}$$

For the case where $f(0) \neq i$, define a new function $F(z)$ by

$$F(z) = \frac{f(z) - \operatorname{Re} f(0)}{\operatorname{Im} f(0)}$$

Then $F(z)$ is an analytic function which maps $\{|z| < 1\}$ to $\{x + iy|y > 0\}$. Moreover we have $F(0) = i$. By previous result, we have

$$\begin{aligned} & |F'(0)| \leq 2 \\ \implies \quad & \left|\frac{f'(0)}{\operatorname{Im} f(0)}\right| \leq 2 \\ \implies \quad & |f'(0)| \leq 2\operatorname{Im} f(0) \end{aligned}$$